Name of college - S. S. College, J-Bad
Deptt - Mathematics
Topic - Indeterminate Forms
(Calculus)
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Indeterminate Form

If $f(x) = \dfrac{g(x)}{\psi(x)}$

and $f(a) = \dfrac{g(a)}{\psi(a)}$ becomes of the form $\dfrac{0}{0}$ - then this form is called indeterminate form.

Also if $f(a) = \dfrac{g(a)}{\psi(a)}$ becomes of the form $\dfrac{\infty}{\infty}$

Thus $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are two indeterminate forms. The other types of indeterminate forms are

$0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0

Of all these form $\dfrac{0}{0}$ is considered to be the Fundamental Indeterminate form, for other types of indeterminate form $\dfrac{0}{0}$ is Calculated by using L-Hospital Rule.

Additional Requirement →

Expansion of some important functions

$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots$

$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots$

$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \cdots$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \cdots$

<u>Method</u> for finding the limit using L-Hospital form

If $f(x) = \frac{g(x)}{\psi(x)}$ if $\frac{g(0)}{\psi(0)} = \frac{0}{0}$

then we shall go on differentiating successively the N^r $g(x)$ and D^r $\psi(x)$ separately and each time we put $x=0$ in the differential co-efficient this process will continue till the form $\frac{0}{0}$ stops to occur.

Ex 1 → Evaluate

$$\lim_{x \to 0} \frac{a^x - 1}{x}$$

$$= \lim_{x \to 0} \frac{e^{x\log a} - 1}{x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^{x\log a} \cdot \log a - 0}{1} = \frac{e^0 \cdot \log a}{1} = \log a$$

Ex2. Evaluate

$$\lim_{x\to 0} \frac{\log(1+kx^2)}{1-\cos x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+kx^2}\cdot 2kx}{\sin x}$$

$$= \lim_{x\to 0} \frac{2kx}{(1+kx^2)\sin x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{2k}{(1+kx^2)\cos x + 2kx\cdot \cos x}$$

$$= \frac{2k}{(1+0)\cdot 1 + 0} = 2k.$$

Evaluate 3

$$\lim_{x\to 0} \frac{x\cdot e^x - \log(1+x)}{x^2} \quad \left[\frac{0}{0}\right]$$

$$\lim_{x\to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x}$$

$$= \lim_{x\to 0} \frac{e^x(1+x) + x\cdot e^x(1+x) - 1}{2x(1+x)}$$

$$= \lim_{x\to 0} \frac{e^x + 2xe^x + x^2e^x - 1}{2x + 2x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x + 2\left[e^x + xe^x\right] + x^2e^x + 2xe^x - 0}{2 + 4x}$$

$$= \lim_{x\to 0} \frac{e^x + 2e^x + 2xe^x + x^2e^x + 2xe^x}{2+4x}$$

$$= \lim_{x\to 0} \frac{3e^x + 4xe^x + x^2e^x}{2+4x}$$

$$= \frac{3e^0 + 4\cdot 0\cdot e^0 + 0\cdot e^0}{2 + 4\cdot 0} = \frac{3\times 1}{2} = \frac{3}{2}$$

<u>Evaluate 4</u> $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ $\left[\frac{0}{0}\right]$

Using L-Hospital Rule.

$$\lim_{x\to 0} \frac{e^x - e^{\sin x}\cos x}{1 - \cos x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x - \cos x\, e^{\sin x}\cos x + e^{\sin x}\sin x}{\sin x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{e^x - \left[\cos^2 x\, e^{\sin x}\cdot\cos x + 2\cos x(-\sin x)e^{\sin x}\right] + e^{\sin x}\cos x\sin x + e^{\sin x}\cdot\cos x}{\cos x}$$

$$= \frac{1 - [1 - 0] + 0 + 1}{1} = 1$$

5. Find the values of a and b, so that
$$\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$$

$$= \lim_{x\to 0} \frac{x\left[1+a\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}\cdots\right)\right] - b\left[x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}\cdots\right]}{x^3} = 1$$

$$= \lim_{x\to 0} \frac{[1+a-b]x + \left(\frac{b}{6}-\frac{a}{2}\right)x^3 + \cdots}{x^3} = 1$$

But the limit of this expression is 1 when $x \to 0$

it is only possible when

$1+a-b=0 \qquad \frac{b}{6}-\frac{a}{2} = 1.$

$\Rightarrow a-b=-1 \qquad \cancel{3}b-3a=6$

$\Rightarrow b-a=1 \qquad b-3a=6$

Now $(b-a)-(b-3a) = 1-6 = -5$

$\Rightarrow 2a=-5 \Rightarrow a=-5/2$

$\therefore b = a+1 = -5/2+1 = -3/2$

Ex.6 Evaluate limit

$$\lim_{x\to 0} \frac{\cosh x - \cos x}{x\sin x} \quad \left|\frac{0}{0}\right|$$

$$\lim_{x\to 0} \frac{\quad}{\sin x + x\cos x} \quad \text{(by using L'Hospital Rule)}$$

$$= \lim_{x\to 0} \frac{\cosh x + \cos x}{\cos x + \cos x - x\sin x}$$

$$= \frac{1+1}{2-0} = 1$$

Evaluate $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$ $\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x}$$

$$= \lim_{x\to 0} \frac{(1+x)\cos x - x\sin x - 1}{2x(1+x)}$$

$$= \lim_{x\to 0} \frac{\cos x + x\cos x - x\sin x - 1}{2x + 2x^2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-\sin x + \cos x - x\sin x - \sin x - x\cos x}{2 + 4x}$$

$$= \frac{1}{2}$$

[Problems based on the form $\frac{\infty}{\infty}$]

Problem 1 Evaluate

$$\lim_{x\to 0} \frac{\log(x-a)}{\log(e^x - e^a)}$$

$$= \lim_{x \to a} \frac{\log(x-a)}{\log(e^x - e^a)} \quad \left(\frac{\infty}{\infty}\right)$$

(using L-Hospital Rule)

$$= \lim_{x \to a} \frac{1}{(x-a)} \cdot \frac{1}{\frac{1}{e^x - e^a} \cdot e^x}$$

$$= \lim_{x \to a} \frac{e^x - a^a}{e^x(x-a)} \cdot \left[\frac{0}{0}\right]$$

$$= \lim_{x \to a} \frac{e^x}{e^x(x-a) + e^x}$$

$$= \frac{e^a}{e^a} = 1$$

2. Evaluate-

$$\lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \pi/2} \frac{1}{x - \pi/2} \cdot \frac{1}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\cos^2 x}{x - \pi/2} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to \pi/2} \frac{-2\cos x \sin x}{1} = 0$$